Palm Theory, Random Measures and Stein Couplings

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Outline

- Stein's Method
- Palm Theory
- A General Normal Apprximation Theorem
- Application to Random Measures and Stochastic Geometry
- Application to Stein Couplings

Stein's Lemma

Lemma 1 (Stein, 1960's)

Let W be a random variable and let $\sigma^2 > 0$. Then $W \sim \mathcal{N}(0, \sigma^2)$ if and only if $\mathbb{E}\{\sigma^2 f'(W) - W f(W)\} = 0$ for all bounded absolutely continuous functions f with bounded f'.

Proof.

(i) *Only if*: By integration by parts.

(ii) *If*: It suffices to consider the case $\sigma^2 = 1$. Let $h \in C_B$ and let f_h be the unique C_B^1 solution of

$$f'(w) - wf(w) = h(w) - \mathbb{E}h(Z) \quad \text{where} \quad Z \sim \mathcal{N}(0, 1).$$

Then

$$\mathbb{E}h(W) - \mathbb{E}h(Z) = \mathbb{E}\{f_h'(W) - W f_h(W)\} = 0.$$

This implies $W \sim \mathcal{N}(0, 1)$.

Stein's Method for Normal Approximation

Stein (1972), *Proc. Sixth Berkeley Symposium*

- Let W be such that $\mathbb{E}W = 0$ and $\mathrm{Var}(W) = 1$.
- If W is not distributed $\mathcal{N}(0,1)$, then $\mathbb{E}\{f'(W) - Wf(W)\} \neq 0$.
- How to quantify the discrepancy between $\mathcal{L}(W)$ and $\mathcal{N}(0,1)$?
- Choose f to be a bounded solution, f_h, of

$$f'(w) - wf(w) = h(w) - \mathbb{E}h(Z) \quad \text{(Stein equation)},$$

  where $Z \sim \mathcal{N}(01,)$ and $h \in \mathcal{G}$, a suitable separating class of functions.
- Then $\mathbb{E}h(W) - \mathbb{E}h(Z) = \mathbb{E}\{f_h'(W) - Wf_h(W)\}$.
- The distance induced by $\mathcal{G}$ is defined as

$$\begin{aligned} d_{\mathcal{G}}(W,Z) &:= \sup_{h\in\mathcal{G}} |\mathbb{E}h(W) - \mathbb{E}h(Z)| \\ &= \sup_{h\in\mathcal{G}} |\mathbb{E}\{f_h'(W) - Wf_h(W)\}|. \end{aligned}$$

Separating Classes of Functions

- These separating classes of functions defined are of interest.

$$\begin{aligned} \mathcal{G}_W &:= \{h; |h(u) - h(v)| \le |u - v|, u, v \in \mathbb{R}\}, \\ \mathcal{G}_K &:= \{h; h(w) = 1 \text{ for } w \le x \text{ and } = 0 \text{ for } w > x, x \in \mathbb{R}\}, \\ \mathcal{G}_{TV} &:= \{h; h(w) = I(w \in A), A \text{ is a Borel subset of } \mathbb{R}\}. \end{aligned}$$

- The distances induced by these three separating classes are respectively called the Wasserstein distance, the Kolmogorov distance, and the total variation distance.
- It is customary to denote $d_{\mathcal{G}_W}$, $d_{\mathcal{G}_K}$ and $d_{\mathcal{G}_{TV}}$ respectively by d_W, d_K and d_{TV}.

Approximation by Other Distributions

- Stein's ideas are very general and applicable to approximations by other distributions.
- Examples are Poisson (Chen (1975)), binomial (Stein (1986)), compound Poisson (Barbour, Chen and Loh (1992)), multivariate normal (Barbour (1990), Götze (1991)), Poisson process (Barbour and Brown (1992)), multinomial (Loh (1992)), exponential (Chatterjee, Fulman and Röllin (2011)).
- Stein expanded his method into a definitive theory in the monograph, *Approximate Computation of Expectations*, IMS Lecture Notes Monogr. Ser. 7. Inst. Math. Statist. (1986).

Palm Measures

- Let Γ be a locally compact separable metric space.
- Let Ξ be a random measure on Γ with finite intensity measure Λ, that is, for every Borel subset $A \subset \Gamma$, $\Lambda(A) = \mathbb{E}\Xi(A)$ and $\Lambda(\Gamma) < \infty$.
- For $\alpha \in \Gamma$, there exists a random measure Ξ_α such that

$$\mathbb{E}\left(\int_\Gamma f(\alpha, \Xi)\Xi(d\alpha)\right) = \mathbb{E}\left(\int_\Gamma f(\alpha, \Xi_\alpha)\Lambda(d\alpha)\right)$$

  for $f(\cdot,\cdot) \geq 0$ (or for real-valued $f(\cdot,\cdot)$ for which the expectations exist).
  (Campbell equation)
- Ξ_α is called the Palm measure associated with Ξ at α.

Palm Measures

- If Ξ is a simple point process, the distribution of Ξ_α can be interpreted as the conditional distribution of Ξ given that a point of Ξ has occurred at α.
- If Ξ is a Poisson point process, then $\mathcal{L}(\Xi_\alpha) = \mathcal{L}(\Xi + \delta_\alpha)$ a.e. Λ.
- If $\Lambda(\{\alpha\}) > 0$, then $\Xi(\{\alpha\})$ is a non-negative random variable with positive mean and $\Xi_\alpha(\{\alpha\})$ is a $\Xi(\{\alpha\})$-size-biased random variable.
- Let $Y = \Xi(\{\alpha\}), Y^s = \Xi_\alpha(\{\alpha\})$ and let $\mu = \mathbb{E}\Xi(\{\alpha\})$. The Campbell equation gives

$$\mathbb{E}Yf(Y) = \mu\mathbb{E}f(Y^s)$$

  for all f for which the expectations exist. This implies that

$$\nu_s(dy) = \frac{y}{\mu}\nu(dy).$$

- In general, we may interpret the Palm measure as a "size-biased random measure".

Normal approximation for random measures

- It has applications to stochastic geometry as many problems therein can be formulated in terms of random measures.
- By taking f to be absolutely continuous from $\mathbb{R}$ to $\mathbb{R}$ such that f' is bounded, the Campbell equation implies

$$\mathbb{E}|\Xi|f(|\Xi|) = \mathbb{E}\int_\Gamma f(|\Xi_\alpha|)\Lambda(d\alpha), \text{ where } |\Xi| = \Xi(\Gamma).$$

- Assume that Ξ and Ξ_α, $\alpha \in \Gamma$, are defined on the same probability space.

Normal approximation for random measures

- Let $B^2 = \mathrm{Var}(|\Xi|)$, and define

$$W = \frac{|\Xi| - \lambda}{B}, \quad W_\alpha = \frac{|\Xi_\alpha| - \lambda}{B},$$

and let

$$\Delta_\alpha = W_\alpha - W.$$

- Then

$$\begin{aligned} \mathbb{E}Wf(W) &= \frac{1}{B}\mathbb{E}\int_\Gamma [f(W_\alpha) - f(W)]\Lambda(d\alpha) \\ &= \mathbb{E}\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt \end{aligned}$$

where

$$\hat{K}(t) = \frac{1}{B}\int_\Gamma [I(\Delta_\alpha > t > 0) - I(\Delta_\alpha < t \leq 0)]\Lambda(d\alpha).$$

Normal approximation for random measures

- Let f_x be the bounded unique solution of the Stein equation

$$f'(w) - f(w) = I(w \leq x) - P(Z \leq x),\ Z \sim \mathcal{N}(0,1).$$

- Then

$$\begin{aligned} d_K(W,Z) &= \sup_{x \in \mathbb{R}} |P(W \leq x) - P(Z \leq x)| \\ &= \sup_{x \in \mathbb{R}} |\mathbb{E}\{f_x'(W) - W f_x(W)\}| \\ &= \sup_{x \in \mathbb{R}} \Big|\mathbb{E}\{f_x'(W) - \int_{-\infty}^{\infty} f_x'(W+t)\hat{K}(t)dt\}\Big|. \end{aligned}$$

A General Theorem

Theorem 2

Let W be such that $\mathbb{E}W = 0$ and $\mathrm{Var}(W) = 1$. Suppose there exists a random function $\hat{K}(t)$ such that

$$\mathbb{E}Wf(W) = \mathbb{E}\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt$$

for all absolutely continuous functions f with bounded f'.

Let $\hat{K}(t) = \hat{K}^{in}(t) + \hat{K}^{out}(t)$ where $\hat{K}^{in}(t) = 0$ for $|t| > 1$. Define $K(t) = \mathbb{E}\hat{K}(t), K^{in}(t) = \mathbb{E}\hat{K}^{in}(t)$, and $K^{out}(t) = \mathbb{E}\hat{K}^{out}(t)$.

Then

$$d_{\mathrm{K}}(W, Z) \leq 2r_1 + 11r_2 + 5r_3 + 10r_4 + 7r_5,$$

where $Z \sim \mathcal{N}(0,1)$.

A General Theorem

In Theorem 2,

$$r_1 = \left[\mathbb{E}\left(\int_{|t|\le 1}(\hat{K}^{in}(t) - K^{in}(t))dt\right)^2\right]^{\frac{1}{2}},$$

$$r_2 = \int_{|t|\le 1} |tK^{in}(t)|dt,$$

$$r_3 = \mathbb{E}\int_{-\infty}^{\infty} |\hat{K}^{out}(t)|dt,$$

$$r_4 = \mathbb{E}\int_{|t|\le 1}(\hat{K}^{in}(t) - K^{in}(t))^2 dt,$$

$$r_5 = \left[\mathbb{E}\int_{|t|\le 1} |t|(\hat{K}^{in}(t) - K^{in}(t))^2 dt\right]^{\frac{1}{2}}.$$

Applications

- Completely random measures.

  A random measure Ξ on Γ is *completely random* if $\Xi(A_1), \cdots, \Xi(A_k)$ are independent whenever $A_1, \cdots, A_k \in \mathcal{B}(\Gamma)$ are pairwise disjoint (Kingman, *Pacific J. Math.* 1967).

- Excursion random measures

$$\Xi(dt) = I((t, X_t) \in E)dt, \;\; E \in \mathcal{B}([0, T] \times \mathcal{S}).$$

- Number of maximal points of a Poisson point process in a region.
- Ginibre-Voronoi tessellation.
- Stein couplings: (G, W', W) such that $\mathbb{E}[Gf(W') - Gf(W)] = \mathbb{E}Wf(W)$ for absolutely continuous f with $f(x) = O(1 + |x|)$. Stein couplings include local dependence, exchangeable pairs, size-bias couplings and others.

Excursion Random Measures

- $\mathcal{S}$ a metric space.
- $\{X_t : 0 \le t \le T\}$ an l-dependent $\mathcal{S}$-valued random process, that is, $\{X_t : 0 \le t \le a\}$ and $\{X_t : b \le t \le T\}$ are independent if $b - a > l$.
- Define the excursion random measure

$$\Xi(dt) = I((t, X_t) \in E)dt, \;\; E \in \mathcal{B}([0,T] \times \mathcal{S})$$

Theorem 3

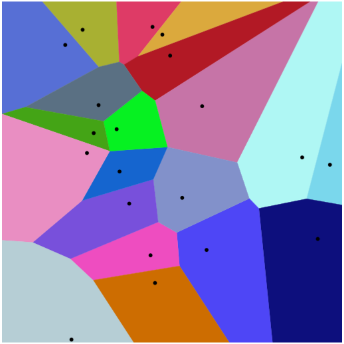
Let $\mu = \mathbb{E}\Xi([0,T])$*,* $B^2 = \mathrm{Var}(|\Xi|)$ *and* $W = \dfrac{|\Xi| - \mu}{B}$*. We have*

$$d_{\mathrm{K}}(W, Z) = O\left(\frac{l^{3/2}\mu^{1/2}}{B^2} + \frac{l^2 \mu}{B^3} \right),$$

where $Z \sim \mathcal{N}(0,1)$*.*

Ginibre-Voronoi Tessellation

- Points are randomly scattered on a plane.
- Cells are formed by drawing lines symmetrically between every two adjacent points.
- Voronoi diagram (20 points and their Voronoi cells)



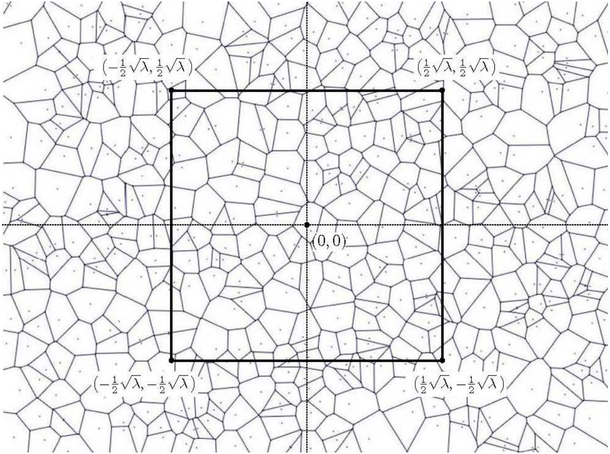
- A lot of interest in the literature in the total edge length of the cells (or of the tessellation).

Ginibre-Voronoi Tessellation

- Assume that the points are from a Ginibre point process.
- The Ginibre point process has attracted considerable attention recently because of its wide use in mobile networks and the Ginibre-Voronoi tesselation.
- The Ginibre point process is a determinantal process and it exhibits repulsion between points.
- The repulsive character makes the cells more regular than those coming from a Poisson point process.
- In applications, the Ginibre-Voronoi tessellation often fits better than the Poisson-Voronoi tessellation.
- It has many applications, such as to biology, epidemiology, aviation and robotic navigation.

Ginibre-Voronoi Tessellation

- Consider the total edge length Y in the square $Q_\lambda = \{(x, y) : -\frac{1}{2}\sqrt{\lambda} \leq x, y \leq \frac{1}{2}\sqrt{\lambda}\}$.



- Interested in how Y behaves as $\lambda \longrightarrow \infty$.

Ginibre-Voronoi Tessellation

Theorem 4

Let Y be the total edge length of the Ginibre-Voronoi tessellation in $Q_\lambda = \{(x, y) : -\frac{1}{2}\sqrt{\lambda} \le x, y \le \frac{1}{2}\sqrt{\lambda}\}$ (excluding edges of infinite length). Let $B^2 = \mathrm{Var}(Y)$ and let $W = \dfrac{Y - \mathbb{E}Y}{B}$.
We have

$$0 < \lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}Y < \infty, \quad 0 < \lim_{\lambda\to\infty} \lambda^{-1}B^2 < \infty,$$

and

$$d_{\mathrm{K}}(W, Z) = O(\frac{\log \lambda}{\lambda^{1/2}}),$$

where $Z \sim \mathcal{N}(0, 1)$.

Poisson functionals

- Peccati (2011) *arXiv:1112.5051*, and others have done work on normal and Poisson approximations for functionals of Possion processes using Stein's method and Malliavin calculus for Poisson processes.
- Döbler and Peccati (2017), *Ann. Probab, to appear*, Döbler, Vidotto and Zheng (2017), *preprint* have proved fourth moment theorems for Poisson Wiener chaos similar to that for Gaussian Wiener chaos.

  If F belongs the the kth Wiener chaos of the standard Brownian motion such that $\mathbb{E}F^2 = 1$, where $k \geq 2$, then

  $$d_{TV}(F, Z) \leq 2\sqrt{\frac{k-1}{3k}}\sqrt{\mathbb{E}F^4 - 3}.$$

  (Nourdin and Peccati (2009), *Probab. Theory Relat. Fields*)
- Peccati et al have also applied their results to stochastic geometry.

A Comparison

- We use Palm theory and our approximations are for random measures, which are more general than point processes and which also cover point processes more general or different from Poisson processes.
- However, by exploiting the special properties of the Poisson process and and the power of Malliavin calculus, Peccati et al are able to do detailed anaylisis of the approximations.
- A possible future direction is to combine Stein's method, Malliavin calculus and Palm theory to study problems in random measures.
- Some encouraging findings in this direction can be found in Döbler and Peccati (2017), Döbler, Vidotto and Zheng (2017).

Stein Couplings

Concept of Stein couplng introduced in Chen and Röllin (2010), arXiv:1003.6039v

- Stein coupling is a general formulation of problems to which Stein's method for norml apprximation is applicable.
- A triple of random variables (W, W', G) defined on the same probability space is called a *Stein coupling* if

$$\mathbb{E}[Gf(W') - Gf(W)] = \mathbb{E}[Wf(W)]$$

  for all absolutely continuous functions f with $f(w) = O(1 + |w|)$.
- By taking $f(w) = 1$, the defining equation implies that $\mathbb{E}W = 0$.
- By taking $f(w) = w$, $\mathrm{Var}(W) = \mathbb{E}[G(W' - W)]$.
- We assume that $\mathrm{Var}(W) = 1$.

Stein Couplings

- Let (W, W', G) be a Stein coupling with $\mathrm{Var}(W) = 1$, that is,

$$\mathbb{E}\left[Gf(W') - Gf(W)\right] = \mathbb{E}Wf(W)$$

for all absolutely continuous f with $f(w) = O(1 + |w|)$.

- Let $\Delta = W' - W$ and let $\mathcal{F}$ be a σ-algebra w.r.t. which W is measurable.
- Then

$$\mathbb{E}Wf(W) = \mathbb{E}\int_{-\infty}^{\infty} f'(W + t)\hat{K}(t)dt,$$

where

$$\hat{K}(t) = \mathbb{E}\{G[I(\Delta > t > 0) - I(\Delta < t \leq 0)]\big|\mathcal{F}\}.$$

Local Dependence

- Suppose $X_1, \cdots, X_n$ are LD1 locally dependent random variables with dependency neighborhoods $B_i \subset \{1, \cdots, n\}$, $i = 1, \cdots, n$, that is, for each i, X_i is independent of $\{X_j : j \in B_i^c\}$.
- Let $W = \sum_{i=1}^n X_i$. Assume that for each i, $\mathbb{E}X_i = 0$ and that $\mathrm{Var}(W) = 1$.
- Define I to be uniformly distributed over $\{1, \cdots, n\}$ and be independent of $X_1, \cdots, X_n$.
- Let $W' = W - \sum_{j \in B_I} X_j$ and $G = -nX_I$.
- Then $\mathbb{E}[Gf(W') - Gf(W)] = \mathbb{E}[nX_I f(W)] = \mathbb{E}Wf(W)$.
- This implies that (W, W', G) is a Stein coupling.

Exchangeable Pairs

Notion of an exchangeable pair introduced by Stein (1986).

- Let (W, W') be an exchangeable pair of random variables, that is, $\mathcal{L}(W, W') = \mathcal{L}(W', W)$, such that $\mathrm{Var}(W) = 1$.
- Suppose there exists a constant $\lambda > 0$ such that

$$\mathbb{E}(W' - W|W) = -\lambda W.$$

- Let f be an absolutely continuous function such that $f(w) = O(1 + |w|)$. Since the function $(w, w') \longmapsto (w' - w)(f(w') + f(w))$ is anti-symmetric, the exchangeability of (W, W') implies

$$\mathbb{E}[(W' - W)(f(W') + f(W))] = 0.$$

- From this, we obtain

$$\mathbb{E}[(W' - W)(f(W') - f(W))] = 2\lambda \mathbb{E} W f(W).$$

- This implies that $(W, W', \frac{1}{2\lambda}(W' - W))$ is a Stein coupling.

Examples

Sums of independent random variables

- Let $X_1, \cdots, X_n$ be independent random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}(\sum_{i=1}^n X_i) = 1$.
- Let $X_1', \cdots, X_n'$ be an independent copy of $X_1, \cdots, X_n$ and let I be independent of $X_1, \cdots, X_n, X_1', \cdots, X_n'$ and be uniformly distributed over $\{1, \cdots, n\}$.
- Define $W = \sum_{i=1}^n X_i$ and $W' = W - X_I + X_I'$.
- Then (W, W') is an exchange pair.
- $\mathbb{E}(W' - W|W) = \mathbb{E}(-X_I + X_I'|W) = -\frac{1}{n}W$.

Examples

Anti-voter model on a complete finite graph

- Let V be a complete finite graph with $|V| = n$ and let $X^{(t)} = \left\{X_i^{(t)}\right\}_{i \in V}, t = 0, 1, \ldots$, where $X_i^{(t)} = +1$ or -1.
- From time t to $t+1$, choose a random vertex i and a random neighbor j of it; then set $X_i^{(t+1)} = -X_j^{(t)}$ and $X_k^{(t+1)} = X_k^{(t)}$ for all $k \neq i$. Start with the stationary distribution.
- Define $U^{(t)} = \sum_{i \in V} X_i^{(t)}$ and $W^{(t)} = \dfrac{U^{(t)}}{\sigma}$ where $\sigma^2 = \mathrm{Var}(U^{(t)})$.
- It is shown in Rinott and Rotar (*Ann. Appl. Probab.* 1997) that $(W^{(t)}, W^{(t+1)})$ is an exchangeable pair and that

$$E(W^{(t+1)} - W^{(t)} | W^{(t)}) = -\frac{2}{|V|} W^{(t)}.$$

Size-bias Couplings

- Let V be non-negative with mean μ and variance σ^2.
- There exists $V^s \geq 0$, called V-size-biased, such that

$$\mathbb{E}Vf(V) = \mu\mathbb{E}f(V^s)$$

  for all f for which the expectations exist.
- Assume that V and V^s are defined on the same probability space and let

$$W = \frac{V-\mu}{\sigma},\ W' = \frac{V^s-\mu}{\sigma},\ G = \frac{\mu}{\sigma}.$$

- Then

$$\mathbb{E}[Gf(W') - Gf(W)] = \mathbb{E}Wf(W).$$

- This shows that (W, W', G) is a Stein coupling.

Additive Functionals of Classical Occupancy Scheme

- Suppose m balls are independently distributed among n urns.
- $P(\text{Each ball falls into urn } i) = p_i$, where $\sum_{i=1}^{n} p_i = 1$.
- For $i = 1, \cdots, n$, let ξ_i be the number of balls in urn i and let $\phi_i : \mathbb{N} \longrightarrow \mathbb{R}$ such that $\mathrm{Var}\left(\sum_{i=1}^{n} \phi_i(\xi_i)\right) > 0$. .
- For $i = 1, \cdots, n$, let $\mu_i = \mathbb{E}\phi_i(\xi_i)$, and let $B^2 = \mathrm{Var}\left(\sum_{i=1}^{n} \phi_i(\xi_i)\right)$.
- Consider

$$W = \frac{1}{B} \sum_{i=1}^{n} \left(\phi_i(\xi_i) - \mu_i\right).$$

Additive Functionals of Classical Occupancy Scheme

- For $i = 1, \cdots, n$, (i) if $\xi_i = 0$, define $\xi'_{ij} = \xi_j$ for $j \neq i$; (ii) if $\xi_i \neq 0$, distribute independently the balls in urn i into urns j, $j \neq i$, with probability $p_j/(1-p_i)$, and let ξ'_{ij} be the resulting number of balls in urn j, $j \neq i$.
- $\mathcal{L}(\xi'_{ij} : j \neq i) = \mathcal{L}(\xi_j : j \neq i | \xi_i = 0)$
- ξ_i is indpendent of $\{\xi'_{ij} : j \neq i\}$.
- Define $G_i = \phi(\xi_i) - \mu_i$ and $W_i = \dfrac{1}{B} \sum_{j \neq i} \big(\phi(\xi'_{ij}) - \mu_j\big)$.
- Then $\mathbb{E}\left[G_i f(W_i) - G_i f(W)\right] = -\mathbb{E} G_i f(W)$.
- Let $I \sim \mathcal{U}\{1, \cdots, n\}$ and be independent of all the other r.v.'s.
- Define $G = -nG_I$ and $W' = W_I$.
- Then $\mathbb{E}\left[G f(W') - G f(W)\right] = \mathbb{E} W f(W)$, that is, (W, W', G) is a Stein coupling.

Additive Functionals of Classical Occupancy Scheme

Theorem 5

Suppose there exist positive constants K_1, K_2 *and* K_3 *such that (i)* $|\phi_i(x)| \leq K_1 e^{K_1 x}$, $x \geq 0$, $i = 1, \cdots, n$, *(ii)* $\max_{1\leq i\leq n} p_i \leq \frac{K_2}{m}$, *and (iii)* $n \leq K_3 m$. *Then there exists a constant* $C = C(K_1, K_2, K_3)$ *such that*

$$d_K(W, Z) \leq C\left(\frac{n^{1/2}}{B^2} + \frac{n}{B^3}\right),$$

where $Z \sim \mathcal{N}(0, 1)$.

Corollary 6

If $\phi_i = \phi$ *and* $p_i = 1/n$, $i = 1, \cdots, n$, *and if* $m \asymp n$, *then* $B^2 \asymp n$ *and*

$$d_K(W, Z) \leq \frac{C}{n^{1/2}}.$$

Thank You

Ginibre point process

We say the point process $\mathbf{X}$ on the complex plane $\mathbb{C}$ ($\cong \mathbb{R}^2$) is the Ginibre point process if its factorial moment measures are given by

$$\nu^{(n)}(dx_1, \ldots, dx_n) = \rho^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n, \ n \geq 1,$$

where $\rho^{(n)}(x_1, \ldots, x_n)$ is the determinant of the $n \times n$ matrix with (i,j)th entry

$$K(x_i, x_j) = \frac{1}{\pi} e^{-\frac{1}{2}(|x_i|^2 + |x_j|^2)} e^{x_i \bar{x}_j}.$$

Here $\bar{x}$ and $|x|$ are the complex conjugate and modulus of x respectively.